

# Quantum Heat Traces

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We study new invariants of elliptic partial differential operators acting on sections of a vector bundle over a closed Riemannian manifold that we call the relativistic heat trace and the quantum heat traces. We obtain some reduction formulas expressing these new invariants in terms of some integral transforms of the usual classical heat trace and compute the asymptotics of these invariants. The coefficients of these asymptotic expansion are determined by the usual heat trace coefficients (which are locally computable) as well as by some new global invariants.

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# 1 Introduction

The heat kernel is one of the most important tools of global analysis, spectral geometry, differential geometry and mathematical physics [15, 11, 13, 16], in particular, quantum field theory, even financial mathematics [7]. In quantum field theory the main objects of interest are described by the Green functions of self-adjoint elliptic partial differential operators on manifolds and their spectral invariants such as the functional determinants. In spectral geometry one is interested in the relation of the spectrum of natural elliptic partial differential operators to the geometry of the manifold. There are also non-trivial links between the spectral invariants and the non-linear completely integrable evolution systems, such as Korteweg-de Vries hierarchy (see, e.g. [16, 8, 4, 9]). In many interesting cases such systems are, in fact, infinite-dimensional Hamiltonian systems, and the spectral invariants of a linear elliptic partial differential operator are nothing but the integrals of motion of this system. In financial mathematics the behavior of the derivative securities (options) is determined by some deterministic parabolic partial differential equations of diffusion type with an elliptic partial differential operator of second order. The conditional probability density is then nothing but the fundamental solution of this equation, in other words, the heat kernel [7].

Instead of studying the spectrum of a differential operator directly one usually studies its spectral functions, that is, spectral traces of some functions of the operator, such as the zeta function, and the heat trace. Usually one does not know the spectrum exactly; that is why, it becomes very important to study various asymptotic regimes. It is well known, for example, that one can get information about the asymptotic properties of the spectrum by studying the short time asymptotic expansion of the heat trace. The coefficients of this expansion, called the heat trace coefficients (or global heat kernel coefficients), play very important role in spectral geometry and mathematical physics [16, 13].

The existence of non-isometric isospectral manifolds demonstrates that the spectrum alone does not determine the geometry (see, e.g. [10]). That is why, it makes sense to study more general invariants of partial differential operators, maybe even such invariants that are not spectral invariants, that is, invariants that depend not only on the eigenvalues but also on the eigenfunctions, and, therefore, contain much more information about the geometry of the manifold.

The case of a Laplace operator on a compact manifold without boundary is well understood and there is a vast literature on this subject, see [13] and the references therein. In this case there is a well defined local asymptotic expansion of the heat kernel, which enables one to compute its diagonal and then the heat trace

by directly integrating the heat kernel diagonal; this gives all heat trace coefficients. The heat trace asymptotics of laplace type operators have been extensively studied in the literature, and many important results have been discovered. The early developments are summarized in the books [13, 11] with extensive bibliography, see also [4, 3, 5, 6].

We initiate the study of new invariants of second-order elliptic partial differential operators acting on sections of vector bundles over compact Riemannian manifolds without boundary. The long term goal of this project is to develop a comprehensive methodology for such invariants in the same way as the theory of the standard heat trace invariants. We draw a deep analogy between the spectral invariants of elliptic operators and the classical and quantum statistical physics [17]. We consider an elliptic self-adjoint positive partial differential operator  $H$  and its square root,  $\omega = H^{1/2}$ , which is an elliptic self-adjoint positive *pseudo*-differential operator of first order.

In Sec. 2 we motivate the study of the invariants: the *relativistic heat trace*

$$\Theta_r(\beta) = \text{Tr} \exp(-\beta\omega) \quad (1.1)$$

and the *quantum heat traces*

$$\Theta_b(\beta, \mu) = \text{Tr} \{ \exp[\beta(\omega - \mu)] - 1 \}^{-1}, \quad (1.2)$$

$$\Theta_f(\beta, \mu) = \text{Tr} \{ \exp[\beta(\omega - \mu)] + 1 \}^{-1}, \quad (1.3)$$

where  $\text{Tr}$  denotes the standard  $L^2$  trace,  $\beta$  is a positive parameter and  $\mu$  is a (generally, non-positive) parameter. We also introduce the corresponding zeta functions: the *relativistic zeta function*

$$Z_r(s, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} e^{\beta\mu} \Theta_r(\beta) \quad (1.4)$$

and the *quantum zeta functions*

$$Z_{b,f}(s, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \Theta_{b,f}(\beta, \mu). \quad (1.5)$$

We show that these new invariants can be reduced to some integrals of the well known classical heat trace

$$\Theta(t) = \text{Tr} \exp(-tH) \quad (1.6)$$

and compute the asymptotics of these invariants as  $\beta \rightarrow 0$ .

In Sec. 3 we review the standard theory of the heat kernel of the Laplace type operator  $H$  in a form suitable for our analysis. To be precise, we consider a closed Riemannian manifold  $M$  of dimension  $n$ , a vector bundle  $\mathcal{V}$  over  $M$  and an elliptic self-adjoint second-order *partial* differential operator  $H$  with a positive definite scalar leading symbol of Laplace type acting on sections of the bundle  $\mathcal{V}$ . We introduce a function  $A_q$  of a complex variable  $q$  defined by

$$A_q = (4\pi)^{n/2} \frac{1}{\Gamma(-q)} \int_0^\infty dt t^{-q-1+n/2} \Theta(t). \quad (1.7)$$

Then we show that for a positive operator  $H$  the function  $A_q$  is entire and its values at non-negative integer points  $q = k$  are equal to the standard heat trace coefficients, which are locally computable, while the values of the function  $A_q$  at half-integer points  $q = k + 1/2$  as well as the values of its derivative at the positive integer points are new global invariants that are not locally computable.

In Sec. 4 we compute the relativistic zeta function  $Z_r(s, \mu)$  and the asymptotics of the relativistic heat trace  $\Theta_r(\beta)$  as  $\beta \rightarrow 0$ . We obtained in even dimension  $n = 2m$ ,

$$\Theta_r(\beta) \sim \sum_{k=0}^{\infty} \beta^{2k-2m} b_k^{(1)} A_k + \sum_{k=0}^{\infty} \beta^{2k+1} b_k^{(2)} A_{k+m+1/2}, \quad (1.8)$$

and in odd dimension  $n = 2m + 1$ ,

$$\Theta_r(\beta) \sim \sum_{k=0}^{\infty} \beta^{2k-2m-1} b_k^{(3)} A_k + \log \beta \sum_{k=0}^{\infty} \beta^{2k+1} b_k^{(4)} A_{k+m+1} + \sum_{k=0}^{\infty} \beta^{2k+1} b_k^{(5)} A'_{k+m+1}, \quad (1.9)$$

and computed all numerical coefficients  $b_k^{(i)}$ .

In Secs. 5 and 6 we express the quantum heat traces in terms of the classical one and compute their asymptotics as  $\beta \rightarrow 0$ . For  $\mu = 0$  we obtain an asymptotic expansion as  $\beta \rightarrow 0$ : in even dimension  $n = 2m$ ,

$$\Theta_f(\beta, 0) \sim \sum_{k=0}^m \beta^{2k-2m} c_k^{(1)} A_k + \sum_{k=0}^{\infty} \beta^{2k+1} c_k^{(2)} A_{k+m+1/2}, \quad (1.10)$$

$$\Theta_b(\beta, 0) = \sum_{k=0}^m \beta^{2k-2m} c_k^{(3)} A_k + \sum_{k=-1}^{\infty} \beta^{2k+1} c_k^{(4)} A_{k+m+1/2}, \quad (1.11)$$

and in odd dimension  $n = 2m + 1$ ,

$$\Theta_f(\beta, 0) \sim \sum_{k=0}^{\infty} \beta^{2k-2m-1} c_k^{(5)} A_k + \log \beta \sum_{k=0}^{\infty} \beta^{2k+1} c_k^{(6)} A_{k+m+1} + \sum_{k=0}^{\infty} \beta^{2k+1} c_k^{(7)} A'_{k+m+1}, \quad (1.12)$$

$$\Theta_b(\beta, 0) = \sum_{k=0}^{m-1} \beta^{2k-2m-1} c_k^{(8)} A_k + \log \beta \sum_{k=-1}^{\infty} \beta^{2k+1} c_k^{(9)} A_{k+m+1} + \sum_{k=-1}^{\infty} \beta^{2k+1} c_k^{(10)} A'_{k+m+1}. \quad (1.13)$$

and computed all numerical coefficients  $c_k^{(i)}$ .

## 2 Quantum Heat Traces

We are going to introduce some new invariants of elliptic operators on manifolds. We will draw a deep analogy from physics. In statistical physics [17] one considers a statistical ensemble of  $N$  *identical* particles. The particles can be in states with a discrete set of energy levels  $\{\omega_k\}_{k=1}^{\infty}$  bounded below,  $\omega_k \geq \omega_1$ ; we assume that the energy levels form an increasing unbounded sequence of real numbers. The state with the lowest energy  $\omega_1$  is called the *ground (or vacuum) state*. Without loss of generality one can always assume that the vacuum energy is positive, i.e.  $\omega_1 > 0$ .

Let  $n_k$  be the average number of particles in the state with the energy  $\omega_k$ . If the ensemble is in *thermodynamical equilibrium* then the number of particles  $n_k$  with the energy  $\omega_k$  is determined by just two parameters: the temperature  $T$  and the chemical potential  $\mu$ . In a particular type of systems, such as photon gas, when the number of particles is not conserved the chemical potential is exactly equal to zero,  $\mu = 0$ . In classical physics, when the particles are *distinguishable*, this number is determined by the *Boltzman distribution*

$$n_k = \exp[-\beta(\omega_k - \mu)], \quad (2.1)$$

where, following standard notation,  $\beta = 1/T$  is the inverse temperature.

Of course, the total number of particles is

$$N(\beta, \mu) = \sum_{k=1}^{\infty} \exp[-\beta(\omega_k - \mu)], \quad (2.2)$$

which is usually taken as the implicit definition of the chemical potential as a function of the temperature and the number of particles

$$\mu(\beta, N) = -\frac{1}{\beta} \log \left\{ \frac{1}{N} \sum_{k=1}^{\infty} \exp(-\beta \omega_k) \right\}. \quad (2.3)$$

The energy of a non-relativistic classical particle of mass  $m$  is described by the *classical Hamiltonian*

$$\omega_c = \frac{1}{2m} H_0, \quad (2.4)$$

where  $H_0$  is an elliptic self-adjoint second-order partial differential operator with positive definite leading symbol of Laplace type acting on sections of a vector bundle over a closed manifold of dimension  $n$ . The eigenvalues  $\lambda_k$  of the operator  $H_0$  play the role of the square of momenta; they are bounded from below. Thus, the total number of classical particles is given by

$$\text{Tr} \exp[-\beta(\omega_c - \mu)] = e^{\beta\mu} \Theta_0 \left( \frac{\beta}{2m} \right), \quad (2.5)$$

where

$$\Theta_0(t) = \text{Tr} \exp(-tH_0) \quad (2.6)$$

is the well-known *classical heat trace*.

The energy of a relativistic classical particle is described by an elliptic self-adjoint first-order pseudo-differential operator (that we call the *relativistic Hamiltonian*)

$$\omega = H^{1/2}, \quad (2.7)$$

where

$$H = H_0 + m^2. \quad (2.8)$$

In order for this to make sense we will always assume that the mass parameter is large enough so that the operator  $H$  is positive. Of course, particles with low momenta and large mass, when  $m^2 \gg \lambda_k$ , are non-relativistic with the energy <sup>2</sup>

$$m + \omega_c = m + \frac{1}{2m} H_0. \quad (2.9)$$

Therefore, the total number of relativistic classical particles is given by

$$\text{Tr} \exp[-\beta(\omega - \mu)] = e^{\beta\mu} \Theta_r(\beta), \quad (2.10)$$

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<sup>2</sup>Strictly speaking, this Hamiltonian only describes the energy levels of the particles with low momenta when the eigenvalues of the operator  $H_0$  are smaller than  $m^2$ .

where

$$\Theta_r(t) = \text{Tr} \exp(-t\omega). \quad (2.11)$$

is the trace of an elliptic pseudo-differential positive operator of first order (that we call *relativistic heat trace*).

Contrary to the classical case, quantum particles are *indistinguishable*. As a matter of fact, there are two kinds of quantum particles, *bosons* and *fermions*. The number of bosons with energy  $\omega_k$  is given by the *Bose-Einstein distribution*

$$n_{b,k} = \frac{1}{\exp[\beta(\omega_k - \mu)] - 1}. \quad (2.12)$$

Notice also that this distribution function is well defined for all energies only if  $\mu < \omega_1$  since  $n_{b,1}$  diverges as  $\mu \rightarrow \omega_1$ . Obviously, particles with large energies, more precisely, when  $\exp[\beta(\omega_k - \mu)] \gg 1$ , are still distributed according to the classical Boltzman distribution.

The total number of particles is now

$$N_b(\beta, \mu) = \sum_{k=1}^{\infty} \frac{1}{\exp[\beta(\omega_k - \mu)] - 1}. \quad (2.13)$$

This again implicitly defines the chemical potential  $\mu = \mu(\beta, N)$  as a function of the temperature and the number of particles. Of course, if we *fix the number of particles*  $N$  and vary the temperature  $\beta$  then the chemical potential is only a function of the temperature,  $\mu = \mu(\beta)$ . One can show [17] that in this case there must exist a critical temperature  $\beta_c$  such that at large temperature,  $\beta < \beta_c$ , the number of particles in the vacuum,  $n_{b,1}(\beta)$ , is negligible. On another hand, at low temperatures,  $\beta > \beta_c$ , the chemical potential  $\mu(\beta)$  approaches  $\omega_1$  with an increasing number of particles in the ground state. In the limit of zero temperature,  $\beta \rightarrow \infty$ , all particles will be in the ground state. The critical temperature  $\beta_c$  is implicitly defined by the equation

$$N_b = \sum_{k=2}^{\infty} \frac{1}{\exp[\beta_c(\omega_k - \omega_1)] - 1}, \quad (2.14)$$

where the summation goes over all states except the ground state. This leads to the so-called *Bose-Einstein condensation* of quantum particles in the ground state with energy  $\omega_1$ .

The fermions obey the *Pauli exclusion principle*, which simply states that no two fermions can be in the same quantum state. This leads to a different energy distribution, so called *Fermi-Dirac distribution*

$$n_{f,k} = \frac{1}{\exp[\beta(\omega_k - \mu)] + 1}. \quad (2.15)$$

Obviously, the Fermi-Dirac distribution also becomes the Boltzman distribution for large energy particles. Contrary to the Bose-Einstein distribution the Fermi-Dirac distribution is defined for any chemical potential. Notice that the number of fermions  $n_{f,k}$  in any state is always less than 1. It is a monotonically decreasing function of energy. Moreover, at zero temperature, as  $\beta \rightarrow \infty$ , the distribution function approaches the step function

$$n_{f,k}(\beta, \mu) \sim \theta(\mu - \omega_k) = \begin{cases} 1, & \text{if } \omega_k < \mu, \\ \frac{1}{2}, & \text{if } \omega_k = \mu, \\ 0, & \text{if } \omega_k > \mu. \end{cases} \quad (2.16)$$

The total number of particles is now

$$N_f(\beta, \mu) = \sum_{k=0}^{\infty} \frac{1}{\exp[\beta(\omega_k - \mu)] + 1}. \quad (2.17)$$

Therefore, at zero temperature the total number of fermions is simply equal to the number  $\mathcal{N}(\mu)$  of quantum states with energy less than  $\mu$ , that is, as  $\beta \rightarrow \infty$ ,

$$N_f(\beta, \mu) \sim \mathcal{N}(\mu). \quad (2.18)$$

Thus, for quantum particles we are led to define two types of new invariants, that we call *quantum heat traces*, the bosonic and the fermionic one by

$$\Theta_b(\beta, \mu) = \text{Tr} \{ \exp[\beta(\omega - \mu)] - 1 \}^{-1}, \quad (2.19)$$

$$\Theta_f(\beta, \mu) = \text{Tr} \{ \exp[\beta(\omega - \mu)] + 1 \}^{-1}, \quad (2.20)$$

where  $\omega$  is the relativistic Hamiltonian (2.7).

Here the parameter  $\mu$  is essential and cannot be simply factored out. In this paper we will restrict mostly to the case of positive operator  $\omega$ , that is,  $\omega_1 > 0$ . We will assume that in the bosonic case the chemical potential  $\mu$  is non-positive,



$\mu \leq 0$  and in the fermionic case it can take any real values. For the fermionic case with  $\mu > 0$  we will decompose the operator as follows

$$\Theta_f(\beta, \mu) = \sum_{\omega_k \leq \mu} \frac{1}{\exp[\beta(\omega_k - \mu)] + 1} + \text{Tr}(I - P_\mu) \{\exp[\beta(\omega - \mu)] + 1\}^{-1}, \quad (2.21)$$

where  $P_\mu$  is the projection operator on the bottom of the spectrum with eigenvalues less or equal to  $\mu$ . Note that

$$\text{Tr}(I - P_\mu) \exp(-tH) = \Theta(t) - \sum_{\omega_k \leq \mu} \exp(-t\omega_k^2). \quad (2.22)$$

Note that at zero temperature, as  $\beta \rightarrow \infty$ , the fermionic heat trace becomes simply the *spectral counting function* (up to exponentially small terms)

$$\Theta_f(\beta, \mu) \sim N(\mu), \quad (2.23)$$

where  $N(\mu)$  is the number of eigenvalues of the operator  $\omega$  less or equal than  $\mu$  (when  $\mu$  coincides with an eigenvalue there is a correction term  $1/2$ ). Therefore, it can be considered as the regularized version of the spectral counting function.

It is easy to see that when  $\exp[\beta(\omega_1 - \mu)] \gg 1$ , that is, for particles with large energies, or, more precisely, as  $\mu \rightarrow -\infty$ , both quantum heat traces are determined by the relativistic heat trace

$$\Theta_b(\beta, \mu) \sim \Theta_f(\beta, \mu) \sim e^{\beta\mu} \Theta_r(\beta). \quad (2.24)$$

One can go even further and define, via the Mellin transform, the corresponding *relativistic zeta function*

$$Z_r(s, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} e^{\beta\mu} \Theta_r(\beta) \quad (2.25)$$

and the *quantum zeta functions*

$$Z_{b,f}(s, \mu) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \Theta_{b,f}(\beta, \mu). \quad (2.26)$$

It is not difficult to show that for non-positive  $\mu \leq 0$

$$Z_r(s, \mu) = \text{Tr}(\omega - \mu)^{-s} \quad (2.27)$$

and

$$Z_b(s, \mu) = \zeta(s) Z_r(s, \mu), \quad (2.28)$$

$$Z_f(s, \mu) = (1 - 2^{1-s}) \zeta(s) Z_r(s, \mu). \quad (2.29)$$

The analyticity properties of these zeta functions depend on the asymptotics of the quantum heat traces as  $\beta \rightarrow 0$  and  $\beta \rightarrow \infty$ , which, in turn, depend crucially on the chemical potential  $\mu$ . It is not difficult to see that for non-positive chemical potential these zeta functions are analytic for sufficiently large real part of  $s$ . More precisely, as shown below, as  $\beta \rightarrow 0$  the quantum heat traces behave as  $\beta^{-n}$ , with  $n = \dim M$ , and, therefore, the quantum zeta functions are analytic for  $\operatorname{Re} s > n$ . Then by using the inverse Mellin transform we can express the quantum heat traces in terms of the zeta functions

$$\Theta_{b,f}(\beta, \mu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \beta^{-s} \Gamma(s) Z_{b,f}(s, \mu), \quad (2.30)$$

where  $c > n$ .

### 3 Heat Kernel of Laplace Type Operator

We are going to employ a very useful representation of the classical heat kernel developed in [1] (see also [4, 6, 2, 3, 7]).

Let  $(M, g)$  be a smooth compact Riemannian manifold of dimension  $n$  without boundary, equipped with a positive definite Riemannian metric  $g$ . We denote the local coordinates on  $M$  by  $x^\mu$ , with Greek indices running over  $1, \dots, n$ . The Riemannian volume element is defined as usual by  $d\operatorname{vol} = dx g^{1/2}$ , where  $g = \det g_{\mu\nu}$ . Let  $\mathcal{V}$  be a vector bundle over the manifold  $M$ ,  $\nabla$  be a connection and  $\mathcal{A}_\mu$  be the connection one-form on the bundle  $\mathcal{V}$ . We assume that the vector bundle  $\mathcal{V}$  is equipped with a Hermitian metric. This naturally identifies the dual vector bundle  $\mathcal{V}^*$  with  $\mathcal{V}$ . We assume that the connection  $\nabla$  is compatible with the Hermitian metric on the vector bundle  $\mathcal{V}$ . The connection is given its unique natural extension to bundles in the tensor algebra over  $\mathcal{V}$  and  $\mathcal{V}^*$ , and, using the Levi-Civita connection of the metric  $g$ , to all bundles in the tensor algebra over  $\mathcal{V}$ ,  $\mathcal{V}^*$ ,  $TM$  and  $T^*M$ ; the resulting connection will usually be denoted just by  $\nabla$ .

We denote by  $C^\infty(\mathcal{V})$  the space of smooth sections of the bundle  $\mathcal{V}$ . The fiber inner product  $\langle \cdot, \cdot \rangle$  on the bundle  $\mathcal{V}$  and the fiber trace,  $\operatorname{tr}$ , defines a natural  $L^2$  inner product  $(\cdot, \cdot)$  and the  $L^2$ -trace,  $\operatorname{Tr}$ , using the invariant Riemannian measure

on the manifold  $M$ . The completion of  $C^\infty(\mathcal{V})$  in this norm defines the Hilbert space  $L^2(\mathcal{V})$  of square integrable sections.

Let  $\nabla^*$  be the formal adjoint to  $\nabla$  defined using the Riemannian metric and the Hermitian structure on  $\mathcal{V}$  and let  $Q$  be a smooth Hermitian section of the endomorphism bundle  $\text{End}(\mathcal{V})$ . A Laplace type operator  $H : C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{V})$  is a partial differential operator of the form

$$H = \nabla^* \nabla + Q = -\Delta + Q, \quad (3.1)$$

where  $\Delta$  is the Laplacian which in local coordinates takes the form

$$\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu = g^{-1/2} (\partial_\mu + \mathcal{A}_\mu) g^{1/2} g^{\mu\nu} (\partial_\nu + \mathcal{A}_\nu). \quad (3.2)$$

Thus, a Laplace type operator is constructed from the following three pieces of geometric data: i) a Riemannian metric  $g$  on  $M$ , ii) a connection 1-form  $\mathcal{A}$  on the vector bundle  $\mathcal{V}$  and iii) an endomorphism  $Q$  of the vector bundle  $\mathcal{V}$ . It is not difficult to show that every elliptic second-order partial differential operator with a positive definite scalar leading symbol is of Laplace type and can be put in this form by choosing the appropriate metric, connection and the endomorphism  $Q$ . It is easy to show that the operator  $H$  is an elliptic essentially self-adjoint partial differential operator. It is well known [13] that the spectrum of the operator  $H$  has the following properties: i) the spectrum is discrete, real and bounded from below, ii) the eigenspaces are finite-dimensional, and iii) the eigenvectors are smooth sections of the vector bundle  $\mathcal{V}$  that form a complete orthonormal basis in  $L^2(\mathcal{V})$ . We denote the eigenvalues and the eigenvectors of the operator  $H$  by  $\{\lambda_k, \varphi_k\}_{k \in \mathbb{Z}_+}$ .

The heat semi-group is the family of bounded operators on  $L^2(\mathcal{V})$  for  $t > 0$

$$U(t) = \exp(-tH). \quad (3.3)$$

The kernel of this operator (called the *heat kernel*) is defined by

$$U(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} \varphi_k(x) \otimes \varphi_k^*(x'), \quad (3.4)$$

where each eigenvalue is counted with multiplicity; it satisfies the heat equation

$$(\partial_t + H) U(t) = 0 \quad (3.5)$$

with the initial condition

$$U(0^+; x, x') = \delta(x, x'). \quad (3.6)$$

The heat trace can be expressed as the integral of the heat kernel diagonal

$$\Theta(t) = \text{Tr} \exp(-tH) = \int_M d\text{vol} \, \text{tr} U(t; x, x). \quad (3.7)$$

That is why we are going to study the heat kernel  $U(t; x, x')$  in the neighborhood of the diagonal of  $M \times M$ , when the points  $x$  and  $x'$  are close to each other. We will keep a point  $x'$  of the manifold fixed and consider a small geodesic ball, i.e. a small neighborhood of the point  $x'$ :  $B_\varepsilon(x') = \{x \in M | r(x, x') < \varepsilon\}$ ,  $r(x, x')$  being the geodesic distance between the points  $x$  and  $x'$ . We will take the radius of the ball sufficiently small, so that each point  $x$  of the ball of this neighbourhood can be connected by a unique geodesic with the point  $x'$ . This can be always done if the size of the ball is smaller than the injectivity radius of the manifold,  $\varepsilon < r_{\text{inj}}$ . Let  $\sigma(x, x')$  be the geodetic interval, also called world function, defined as one half the square of the length of the geodesic connecting the points  $x$  and  $x'$

$$\sigma(x, x') = \frac{1}{2} r^2(x, x'). \quad (3.8)$$

The first derivatives of this function with respect to  $x$  and  $x'$ ,  $\sigma_\mu = \nabla_\mu \sigma$  and  $\sigma_{\mu'} = \nabla_{\mu'} \sigma$ , define tangent vector fields to the geodesic at the points  $x$  and  $x'$  respectively pointing in opposite directions and the determinant of the mixed second derivatives defines the so called Van Vleck-Morette determinant

$$\Delta(x, x') = g^{-1/2}(x) \det \left[ -\nabla_\mu \nabla_{\nu'} \sigma(x, x') \right] g^{-1/2}(x'). \quad (3.9)$$

This object should not be confused with the Laplacian. Let, finally,  $\mathcal{P}(x, x')$  denote the parallel transport operator of sections of the vector bundle  $\mathcal{V}$  along the geodesic from the point  $x'$  to the point  $x$ . Here and everywhere below the coordinate indices of the tangent space at the point  $x'$  are denoted by primed Greek letters. They are raised and lowered by the metric tensor  $g_{\mu'\nu'}(x')$  at the point  $x'$ . The derivatives with respect to  $x'$  will be denoted by primed Greek indices as well.

To study the local behavior of the heat kernel we use the following ansatz (motivated by the heat kernel in the Euclidean space)

$$U(t; x, x') = (4\pi t)^{-n/2} \Delta^{1/2}(x, x') \exp\left(-\frac{1}{2t} \sigma(x, x')\right) \mathcal{P}(x, x') \Omega(t; x, x'). \quad (3.10)$$

The function  $\Omega(t; x, x')$ , called the *transport function*, is a section of the endomorphism vector bundle  $\text{End}(V)$  over the point  $x'$ . For the sake of simplicity we will

omit the space variables  $x$  and  $x'$  when it will not cause any confusion. Using the definition of the functions  $\sigma(x, x')$ ,  $\Delta(x, x')$  and  $\mathcal{P}(x, x')$  it is not difficult to find that the transport function satisfies a transport equation

$$\left(\partial_t + \frac{1}{t}D + \tilde{H}\right)\Omega(t) = 0, \quad (3.11)$$

where  $D$  is the radial vector field, i.e. operator of differentiation along the geodesic, defined by

$$D = \sigma^\mu \nabla_\mu, \quad (3.12)$$

and  $\tilde{H}$  is a second-order differential operator defined by (here, again,  $\Delta$  is not the Laplacian but the Van-Vleck-Morette determinant)

$$\tilde{H} = \mathcal{P}^{-1} \Delta^{-1/2} H \Delta^{1/2} \mathcal{P}. \quad (3.13)$$

The initial condition for the transport function is obviously

$$\Omega(0; x, x') = I, \quad (3.14)$$

where  $I$  is the identity endomorphism of the vector bundle  $\mathcal{V}$  over  $x'$ . One can show that if the operator  $H$  is positive then the transport function  $\Omega(t)$  satisfies the following asymptotic conditions

$$\lim_{t \rightarrow \infty, 0} t^\alpha \partial_t^N \Omega(t) = 0 \quad \text{for any } \alpha \in \mathbb{R}_+, N \in \mathbb{Z}_+. \quad (3.15)$$

Now, we consider a slightly modified version of the Mellin transform of the function  $\Omega(t)$  introduced in [1]

$$a_q = \frac{1}{\Gamma(-q)} \int_0^\infty dt t^{-q-1} \Omega(t). \quad (3.16)$$

The integral (3.16) converges for  $\operatorname{Re} q < 0$ . By integrating by parts  $N$  times and using the asymptotic conditions (3.15) we also get

$$a_q = \frac{1}{\Gamma(-q + N)} \int_0^\infty dt t^{-q-1+N} (-\partial_t)^N \Omega(t). \quad (3.17)$$

This integral converges for  $\operatorname{Re} q < N - 1$ . Using this representation one can prove that [1] the function  $a_q$  is an entire function of  $q$  satisfying the asymptotic condition

$$\lim_{|q| \rightarrow \infty, \operatorname{Re} q < N} \Gamma(-q + N) a_q = 0, \quad \text{for any } N > 0. \quad (3.18)$$

Moreover, the values of the function  $a_q$  at the integer positive points  $q = k$  are given by the Taylor coefficients

$$a_k = (-\partial_t)^k \Omega(t) \Big|_{t=0}. \quad (3.19)$$

By inverting the Mellin transform we obtain for the transport function

$$\Omega(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, t^q \Gamma(-q) a_q \quad (3.20)$$

where  $c < 0$ , which, by using our ansatz (3.10), immediately gives also the heat trace

$$\Theta(t) = (4\pi)^{-n/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, t^{q-n/2} \Gamma(-q) A_q, \quad (3.21)$$

where

$$A_q = \int_M d\text{vol} \, \text{tr} \, a_q(x, x). \quad (3.22)$$

Note that the global function  $A_q$  is determined by the Mellin transform of the heat trace

$$A_q = (4\pi)^{n/2} \frac{1}{\Gamma(-q)} \int_0^\infty dt \, t^{-q-1+n/2} \Theta(t). \quad (3.23)$$

Substituting this ansatz into the transport equation we get a functional equation for the function  $a_q$

$$\left(1 + \frac{-D}{q}\right) a_q = \tilde{H} a_{q-1}. \quad (3.24)$$

The initial condition for the transport function is translated into

$$a_0 = I. \quad (3.25)$$

For integer  $q = k = 1, 2, \dots$  the functional equation (3.24) becomes a recursion system that, together with the initial condition (3.25), determines all coefficients  $a_k$ .

Thus, we have reduced the problem of solving the heat equation to the following problem: one has to find an entire function of  $q$ ,  $a_q(x, x')$ , that satisfies the functional equation (3.24) with the initial condition (3.25) and the asymptotic

condition (3.18). The function  $a_q$  turns out to be extremely useful in computing the heat kernel, the resolvent kernel, the zeta-function and the determinant of the operator  $H$ . It contains the same information about the operator  $H$  as the heat kernel. In some cases the function  $a_q$  can be constructed just by analytical continuation from the integer positive values  $a_k$  [1]. Now we are going to do the usual trick, namely, to move the contour of integration over  $q$  in (3.20) to the right. Due to the presence of the gamma function  $\Gamma(-q)$  the integrand has simple poles at the non-negative integer points  $q = 0, 1, 2, \dots$ , which contribute to the integral while moving the contour. This gives the asymptotic expansion as  $t \rightarrow 0$

$$\Omega(t) \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k. \quad (3.26)$$

The heat trace has an analogous asymptotic expansion as  $t \rightarrow 0$

$$\Theta(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A_k, \quad (3.27)$$

where

$$A_k = \int_M d\text{vol} \, \text{tr} \, a_k(x, x). \quad (3.28)$$

We can apply our ansatz for computation of the complex power of the operator  $H$  defined by

$$G_s(x, x') = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} U(t; x, x'). \quad (3.29)$$

Outside the diagonal, i.e. for  $x \neq x'$  or  $\sigma \neq 0$ , this integral converges for any  $s$  and defines an entire function of  $s$ . Using our ansatz for the heat kernel one can obtain the so-called *Mellin-Barnes representation* of the function  $G_s$  [1]

$$G_s(x, x') = (4\pi)^{-n/2} \Delta^{1/2} \mathcal{P} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \frac{\Gamma(-q)\Gamma(-q-s+n/2)}{\Gamma(s)} \left(\frac{\sigma}{2}\right)^{q+s-n/2} a_q(x, x') \quad (3.30)$$

where  $c < -\text{Re } s + n/2$ . The integrand in this formula is a meromorphic function of  $q$  with some simple and maybe some double poles. If we move the contour of integration to the right, we get contributions from the simple poles in form of

powers of  $\sigma$  and a logarithmic part due to the double poles (if any). This gives the complete structure of diagonal singularities of  $G_s(x, x')$  (see [2]).

For sufficiently large  $\text{Re } s$ , more precisely,  $\text{Re } s > n/2$  the integral (3.29) converges even on the diagonal, that is, for  $x = x'$ . In this case there are no singularities at all and there is a well defined diagonal limit as  $x \rightarrow x'$

$$G_s(x, x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} \varphi_k(x) \varphi_k^*(x). \quad (3.31)$$

On another hand, by using (3.16) on the diagonal we obtain

$$a_q(x, x) = (4\pi)^{n/2} \frac{\Gamma(-q + n/2)}{\Gamma(-q)} \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{-q+n/2}} \varphi_k(x) \varphi_k^*(x); \quad (3.32)$$

therefore, we obtain a very simple formula

$$G_s(x, x) = (4\pi)^{-n/2} \frac{\Gamma(s - n/2)}{\Gamma(s)} a_{n/2-s}(x, x). \quad (3.33)$$

This gives automatically the zeta-function

$$\zeta_H(s) = \text{Tr } H^{-s} = (4\pi)^{-n/2} \frac{\Gamma(s - n/2)}{\Gamma(s)} A_{n/2-s}. \quad (3.34)$$

Since  $A_q$  is an entire function this shows that the zeta function is a meromorphic function and gives the complete structure of its singularities (simple poles): an infinite sequence at the half-integer points  $s = [n/2] + 1/2 - k$ , ( $k = 0, 1, 2, \dots$ ) for odd dimension and a finite number of poles at integer points  $s = 1, 2, \dots, n/2$  for even dimension. In particular, the zeta-function is analytic at the origin; its value at the origin is given by

$$\zeta_H(0) = \begin{cases} 0 & \text{for odd } n = 2m + 1, \\ (4\pi)^{-m} \frac{(-1)^m}{m!} A_m & \text{for even } n = 2m, \end{cases} \quad (3.35)$$

and its derivative at the origin (which determines the determinant of the operator  $H$ ) is

$$\zeta'_H(0) = \begin{cases} (-1)^{m+1} \pi^{-m} \frac{m!}{(2m+1)!} A_{m+1/2} & \text{for odd } n = 2m + 1, \\ (4\pi)^{-m} \frac{(-1)^m}{m!} \{-A'_m + [\psi(m+1) + \mathbf{C}]A_m\} & \text{for even } n = 2m. \end{cases} \quad (3.36)$$



Here  $\psi(z) = (d/dz) \log \Gamma(z)$  is the psi-function,  $\mathbf{C} = -\psi(1) = 0.577 \dots$  is the Euler constant, and

$$A'_m = \frac{\partial}{\partial q} A_q \Big|_{q=m}. \quad (3.37)$$

Note that the value  $\zeta_H(0)$  is determined by the locally computable heat kernel coefficient  $A_k$  whereas the derivative of the zeta function  $\zeta'_H(0)$  is determined by the global invariants  $A_{m+1/2}$  and  $A'_m$  which are not locally computable.

## 4 Relativistic Heat Trace and Zeta Function

First of all, we compute the relativistic zeta function. We use the integral [18]

$$\exp(-x) = (4\pi)^{-1/2} \int_0^\infty dt t^{-3/2} \exp\left(-\frac{1}{4t} - tx^2\right), \quad (4.1)$$

which is valid for any  $x \geq 0$  and eq. (2.25) to obtain

$$Z_r(s, \mu) = \int_0^\infty dt g(t, s, \mu) \Theta(t), \quad (4.2)$$

where

$$g(t, s, \mu) = (4\pi)^{-1/2} \frac{1}{\Gamma(s)} t^{-3/2} \int_0^\infty d\tau \tau^s \exp\left(-\frac{\tau^2}{4t} + \mu\tau\right). \quad (4.3)$$

Now, by expanding this in powers of  $\mu$  we get

$$g(t, s, \mu) = (4\pi)^{-1/2} \frac{1}{\Gamma(s)} \sum_{k=0}^\infty \frac{\mu^k}{k!} t^{(s+k)/2-1} 2^{s+k} \Gamma[(s+k+1)/2]. \quad (4.4)$$

Further, by using the obvious relation (3.23) we obtain

$$Z_r(s, \mu) = (4\pi)^{-(n+1)/2} \frac{1}{\Gamma(s)} \sum_{k=0}^\infty \frac{\mu^k}{k!} 2^{s+k} \Gamma[(s+k+1)/2] \Gamma[(s+k-n)/2] A_{(n-s-k)/2}. \quad (4.5)$$

This enables one to extend the zeta function  $Z_r(s, \mu)$  to a meromorphic function of  $s$ , in particular, it is analytic at  $s = 0$ . Note that the same result can be obtained by using the expansion

$$Z_r(s, \mu) = \sum_{k=0}^\infty \frac{\mu^k}{k!} \frac{\Gamma(s+k)}{\Gamma(s)} \zeta_H[(s+k)/2] \quad (4.6)$$

and the eq. (3.34). As a result, for  $\mu = 0$  we obtain a very simple formula

$$Z_r(s, 0) = (4\pi)^{-(n+1)/2} 2^s \frac{\Gamma[(s+1)/2] \Gamma[(s-n)/2]}{\Gamma(s)} A_{(n-s)/2}. \quad (4.7)$$

Next, we use the integral (4.1) to reduce the relativistic heat trace to the classical heat trace (2.6), that is,

$$\Theta_r(\beta) = (4\pi)^{-1/2} \int_0^\infty dt t^{-3/2} \exp\left(-\frac{1}{4t}\right) \Theta(t\beta^2). \quad (4.8)$$

Now, by using eqs. (4.8) and (3.21) we can express the relativistic heat trace in terms of the function  $A_q$  via a Mellin-Barnes integral

$$\Theta_r(\beta) = 2(4\pi)^{-(n+1)/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \Gamma(-q) \Gamma[-q + (n+1)/2] \left(\frac{\beta}{2}\right)^{2q-n} A_q, \quad (4.9)$$

It is worth noting the striking resemblance of this equation to the kernel of the function  $G_s$  given by eq. (3.30) by replacing  $\sigma$  by  $\beta^2$ . Therefore, the asymptotics of the relativistic heat kernel  $\Theta_r(\beta)$  as  $\beta \rightarrow 0$  can be computed exactly in the same way as the diagonal singularities of  $G_s(x, x')$  as  $\sigma \rightarrow 0$ . Namely, we move the contour of integration to the right to get contributions from the simple poles in form of powers of  $\beta$  and a logarithmic part,  $\log \beta$ , due to the double poles (if any, depending on the dimension).

Integrals of this type are a particular case of the so-called Mellin-Barnes integrals. They are a very powerful tool in computing the heat trace asymptotics. Since we will use them quite often we prove the following lemma

**Lemma 1** *Let  $f(q)$  be a function of a complex variable  $q$  that is analytic in the right half-plane and decreases sufficiently fast at infinity in the right half-plane. Let  $c < 0$  and  $m$  be a positive integer and  $I(t)$  and  $J(t)$  be two functions of a positive real variable  $t$  defined by*

$$I(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \Gamma(-q) \Gamma(-q + m + 1/2) t^q f(q), \quad (4.10)$$

$$J(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \Gamma(-q) \Gamma(-q + m + 1) t^q f(q). \quad (4.11)$$

Then there are asymptotic expansion as  $t \rightarrow 0$

$$I(t) = I_1(t) + I_2(t) + I_3(t), \quad (4.12)$$

$$J(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t), \quad (4.13)$$

where

$$I_1(t) \sim \sqrt{\pi} \sum_{k=0}^m (-1)^k \frac{(2m-2k)!}{k!(m-k)!} 2^{2k-2m} t^k f(k) \quad (4.14)$$

$$I_2(t) \sim (-1)^m \frac{1}{2} \sqrt{\pi} \sum_{k=0}^{\infty} \frac{k!}{(k+m+1)!(2k+1)!} 2^{2k+2} t^{k+m+1} f(k+m+1) \quad (4.15)$$

$$I_3(t) \sim -(-1)^m \frac{1}{2} \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!(2k+2m+1)!} 2^{2k+2m+2} t^{k+m+1/2} f(k+m+1/2). \quad (4.16)$$

$$J_1(t) \sim \sum_{k=0}^m (-1)^k \frac{(m-k)!}{k!} t^k f(k) \quad (4.17)$$

$$J_2(t) \sim (-1)^m \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} t^{k+m+1} [\psi(k+m+2) + \psi(k+1)] f(k+m+1) \quad (4.18)$$

$$J_3(t) \sim -(-1)^m \log t \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} t^{k+m+1} f(k+m+1) \quad (4.19)$$

$$J_4(t) \sim -(-1)^m \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} t^{k+m+1} f'(k+m+1). \quad (4.20)$$

We would like to stress the difference between different parts of these functions. The functions  $I_1$  and  $J_1$  are just polynomials. The functions  $I_2$  and  $J_2$  contain only integer powers of  $t$  with the coefficients determined by the values of the function  $f$  at integer points. The function  $I_3$  contains *half-integer powers* of  $t$  with the coefficients determined by the values of the function  $f$  at *half-integer points* and the function  $J_3$  is proportional to the *logarithm*  $\log t$ . Finally, the coefficients of the function  $J_4$  are determined by the *derivatives* of the function  $f$  at the integer points.

*Proof.* This lemma can be proved by the residue calculus. The integrand for the function  $I$  has simple poles at the points the integer points  $q = k$  and the

half-integer points  $q = k + m + 1/2$ , with  $(k = 0, 1, 2, \dots)$ . The integrand for the function  $J$  has simple poles at  $q = 0, 1, 2, \dots, m$  and *double* poles at the points  $q = k + m + 1$ . Now, the lemma is proved by using the formula (for an integer  $k \geq 0$  and  $z \rightarrow 0$ ) [12]

$$\Gamma(-k + z) = \frac{(-1)^k}{k!} \left\{ \frac{1}{z} + \psi(k + 1) + O(z) \right\} \quad (4.21)$$

and using the properties of the gamma-function.

We apply this lemma to the Mellin-Barnes representation of the relativistic heat trace (4.9). We decompose the trace according to

$$\Theta_r(\beta) = \Theta_r^{\text{sing}}(\beta) + \Theta_r^{\text{loc}}(\beta) + \Theta_r^{\text{non-loc}}(\beta). \quad (4.22)$$

Now, we need to distinguish the cases of odd and even dimensions. We obtain: in even dimension  $n = 2m$ ,

$$\Theta_r^{\text{sing}}(\beta) \sim \frac{1}{2}(4\pi)^{-m+1/2} \sum_{k=0}^m (-1)^k \frac{(2m-2k)!}{k!(m-k)!} \frac{1}{\beta^{2m-2k}} A_k, \quad (4.23)$$

$$\Theta_r^{\text{loc}}(\beta) \sim (-1)^m \frac{1}{2} (4\pi)^{-m} \sum_{k=0}^{\infty} \frac{k!}{(k+m+1)!(2k+1)!} \beta^{2k+2} A_{k+m+1}, \quad (4.24)$$

$$\Theta_r^{\text{non-loc}}(\beta) \sim -(-1)^m \pi^{-m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!(2k+2m+1)!} \beta^{2k+1} A_{k+m+1/2}, \quad (4.25)$$

and in odd dimension  $n = 2m + 1$ ,

$$\Theta_r^{\text{sing}}(\beta) \sim 2(4\pi)^{-m-1} \sum_{k=0}^m (-1)^k \frac{(m-k)!}{k!} \left( \frac{2}{\beta} \right)^{2m-2k+1} A_k \quad (4.26)$$

$$\begin{aligned} \Theta_r^{\text{loc}}(\beta) \sim & (-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left( \frac{\beta}{2} \right)^{2k+1} \\ & \times \left\{ \psi(k+m+2) + \psi(k+1) - 2 \log(\beta/2) \right\} A_{k+m+1} \end{aligned} \quad (4.27)$$

$$\Theta_r^{\text{non-loc}}(\beta) \sim -(-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left( \frac{\beta}{2} \right)^{2k+1} A'_{k+m+1}. \quad (4.28)$$

Note that the coefficients of the parts  $\Theta_r^{\text{sing}}(\beta)$  and  $\Theta_r^{\text{loc}}(\beta)$  are determined by the heat kernel coefficients  $A_k$  (which are locally computable) whereas the part  $\Theta_r^{\text{non-loc}}(\beta)$  is determined by the values of the function  $A_q$  at the half-integer points  $A_{k+m+1/2}$  and the derivatives at the integer points  $A'_{k+m+1}$ , which are non-locally computable; they are rather related to the values of the zeta function and its derivatives at negative half-integer points  $\zeta_H(-k-1/2)$  and  $\zeta'_H(-k-1/2)$ ,  $k = 0, 1, 2, \dots$

## 5 Reduction of Quantum Heat Traces

Next, we will reduce the quantum heat traces to the classical heat trace as well. Let  $E_f$  and  $E_b$  be functions defined by

$$E_f(x) = \frac{1}{e^x + 1} = \frac{1}{2} \left[ 1 - \tanh\left(\frac{x}{2}\right) \right], \quad (5.1)$$

$$E_b(x) = \frac{1}{e^x - 1} = \frac{1}{2} \left[ \coth\left(\frac{x}{2}\right) - 1 \right], \quad (5.2)$$

Notice that

$$E_f(x) = E_b(x) - 2E_b(2x). \quad (5.3)$$

These functions can be represented as series which converge for any  $x > 0$

$$E_f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} e^{-kx}, \quad (5.4)$$

$$E_b(x) = \sum_{k=1}^{\infty} e^{-kx}, \quad (5.5)$$

Now, by using eqs. (4.1) we can rewrite these functions in the form

$$E_{b,f}[\beta(\omega - \mu)] = \int_0^{\infty} dt h_{b,f}(t, \beta\mu) \exp(-t\beta^2\omega^2), \quad (5.6)$$

where

$$h_f(t, \beta\mu) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=1}^{\infty} (-1)^{k+1} k \exp\left(-\frac{k^2}{4t} + k\beta\mu\right), \quad (5.7)$$

$$h_b(t, \beta\mu) = (4\pi)^{-1/2} t^{-3/2} \sum_{k=1}^{\infty} k \exp\left(-\frac{k^2}{4t} + k\beta\mu\right), \quad (5.8)$$

It is easy to see that these functions satisfy the relations

$$h_f(t, \beta\mu) = h_b(t, \beta\mu) - \frac{1}{2}h_b\left(\frac{t}{4}, \beta\mu\right). \quad (5.9)$$

To compute the asymptotics of the quantum heat traces we will need the asymptotics of the functions  $h_{b,f}(t, \beta\mu)$  as  $t \rightarrow 0$  and  $t \rightarrow \infty$ . First of all, it is easy to see that as  $t \rightarrow 0$

$$h_{b,f}(t, \beta\mu) \sim (4\pi)^{-1/2} t^{-3/2} \exp\left(-\frac{1}{4t} + \beta\mu\right). \quad (5.10)$$

Therefore, the integral (5.21) defining the quantum heat traces converges at  $t \rightarrow 0$ . The asymptotics of these functions as  $t \rightarrow \infty$  (and  $\mu < 0$ ) are computed as follows

$$h_f(t, \beta\mu) \sim (4\pi)^{-1/2} t^{-3/2} \frac{e^{\beta\mu}}{(e^{\beta\mu} + 1)^2}, \quad (5.11)$$

$$h_b(t, \beta\mu) \sim (4\pi)^{-1/2} t^{-3/2} \frac{e^{\beta\mu}}{(e^{\beta\mu} - 1)^2}. \quad (5.12)$$

Therefore, the integral (5.21) converges at  $t \rightarrow \infty$  provided the operator  $H$  is positive.

The case  $\mu = 0$  is more complicated. For the fermionic case the eq. (5.11) also holds for  $\mu = 0$ ; one just takes the limit  $\mu \rightarrow 0^-$  to get the asymptotics as  $t \rightarrow \infty$ ,

$$h_f(t, 0) \sim \frac{1}{8\sqrt{\pi}} t^{-3/2}. \quad (5.13)$$

In the bosonic case the limit  $\mu \rightarrow 0$  is singular. By carefully examining the behavior of the function  $h_b$  we obtain as  $t \rightarrow \infty$ ,

$$h_b(t, 0) \sim \frac{1}{\sqrt{\pi}} t^{-1/2}. \quad (5.14)$$

More generally, by using the Taylor expansions

$$\tanh x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k} - 1)B_{2k}}{(2k)!} x^{2k-1}, \quad (5.15)$$

$$\coth x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!} x^{2k-1}, \quad (5.16)$$

where  $B_k$  are the Bernulli numbers, one can compute the asymptotic expansion of the functions  $h_{b,f}(t, 0)$  as  $t \rightarrow \infty$ ; we obtain

$$h_f(t, 0) \sim (4\pi)^{-1/2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k} - 1)B_{2k}}{2^{2k}k!} t^{-k-1/2}, \quad (5.17)$$

$$h_b(t, 0) \sim \frac{1}{\sqrt{\pi}} t^{-1/2} - (4\pi)^{-1/2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{2k}}{2^{2k-1}k!} t^{-k-1/2}. \quad (5.18)$$

The quantum heat traces now take the form

$$\Theta_{b,f}(\beta, \mu) = \text{Tr } E_{b,f}[\beta(\omega - \mu)]. \quad (5.19)$$

In particular, there is a relation

$$\Theta_f(\beta, \mu) = \Theta_b(\beta, \mu) - 2\Theta_b(2\beta, \mu). \quad (5.20)$$

Thus, by using eqs. (5.19) and (5.6) we obtain for the quantum heat traces

$$\Theta_{b,f}(\beta, \mu) = \int_0^{\infty} dt h_{b,f}(t, \beta\mu) \Theta(t\beta^2), \quad (5.21)$$

which reduces the calculation of the quantum heat traces to the calculation of the classical one (2.6). By using this representation we get for the quantum zeta functions

$$Z_{b,f}(s, \mu) = \int_0^{\infty} dt G_{b,f}(t, s, \mu) \Theta(t), \quad (5.22)$$

where

$$G_{b,f}(t, s, \mu) = \frac{1}{\Gamma(s)} \int_0^{\infty} d\beta \beta^{s-3} h_{b,f}\left(\frac{t}{\beta^2}, \beta\mu\right). \quad (5.23)$$

Another representation of the quantum heat traces can be obtained as follows. Let  $f(z)$  be a function of a complex variable  $z$  such that it is analytic in the region  $\text{Re } z > \gamma$ , with some real parameter  $\gamma$ , and decreases sufficiently fast at infinity in the right half-plane. Then the function  $f(-ip)$  of a complex variable  $p$  is analytic in the region  $\text{Im } p > \gamma$  and decreases sufficiently fast at infinity in the upper half-plane. Let  $C$  be a contour in the upper half-plane of the complex variable  $p$  that goes from  $-\infty + i\gamma$  to  $\infty + i\gamma$  above all singularities of the function  $f(-ip)$ . Then for any  $x > 0$  there holds

$$f(x) = \frac{1}{2\pi i} \int_C dp \frac{2p}{p^2 + x^2} f(-ip); \quad (5.24)$$

with the only singularity above the contour  $C$  being the simple pole at  $p = ix$ . This can also be rewritten in the form

$$f(x) = \int_0^\infty dt h(t) \exp(-tx^2), \quad (5.25)$$

where

$$h(t) = \frac{1}{\pi i} \int_C dp p f(-ip) \exp(-tp^2). \quad (5.26)$$

By using this equation we can represent the functions  $E_{b,f}$  in the form

$$E_{b,f}[\beta(\omega - \mu)] = \frac{1}{2\pi i} \int_C dp \frac{2p}{p^2 + \beta^2 \omega^2} \frac{1}{\exp(-ip - \beta\mu) \mp 1}. \quad (5.27)$$

Here  $C$  is a  $\vee$  shaped contour going from  $e^{i3\pi/4}\infty$  to 0 and then from 0 to  $e^{i\pi/4}\infty$  encircling only one pole at  $i\beta\omega$ . The integrand has a sequence  $\{p_k\}_{k \in \mathbb{Z}}$ , of other simple poles:

$$p_k = 2k\pi + i\beta\mu, \quad (5.28)$$

in the bosonic case and

$$p_k = (2k + 1)\pi + i\beta\mu \quad (5.29)$$

in the fermionic case. Therefore, for  $\mu < 0$  all other poles of the integrand are in the lower half-plane and the contour of integration  $C$  can be deformed just to the real axis (with the integral understood in the *principal value* sense).

Now, by using the integral representation (5.27) we get the same eq. (5.6) with

$$h_{b,f}(t, \beta\mu) = \frac{1}{2\pi i} \int_C dp \frac{2p}{\exp(-ip - \beta\mu) \mp 1} \exp(-tp^2). \quad (5.30)$$

For  $\mu < 0$  this can be written as

$$\begin{aligned} h_f(t, \beta\mu) &= \frac{1}{2\pi} \int_{\mathbb{R}} dp p \tan\left(\frac{p - i\beta\mu}{2}\right) \exp(-tp^2) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} dp \frac{p \sin p}{\cosh(\beta\mu) + \cos p} \exp(-tp^2), \end{aligned} \quad (5.31)$$

$$\begin{aligned} h_b(t, \beta\mu) &= \frac{1}{2\pi} \int_{\mathbb{R}} dp p \cot\left(\frac{p - i\beta\mu}{2}\right) \exp(-tp^2) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} dp \frac{p \sin p}{\cosh(\beta\mu) - \cos p} \exp(-tp^2). \end{aligned} \quad (5.32)$$



These equations also hold in the limiting case of  $\mu \rightarrow 0$ , with the integrals taken in the principal value sense; the imaginary part cancels out.

Note that in the fermionic case none of the poles  $p_k$  lie on the imaginary axis. Therefore, for  $\mu > 0$  in the fermionic case the contour  $C$  goes above all the poles  $p_k$  in the upper left half-plane, goes underneath the pole at  $p = i\omega$ , and then goes again above the poles  $p_k$  in the right upper half-plane. Another important observation is that

$$\operatorname{Re} p_k^2 = (2k+1)^2 \pi^2 - \beta^2 \mu^2, \quad (5.33)$$

and, therefore, for sufficiently small  $\beta\mu$ , more precisely, for  $\beta\mu < \pi$ , the real part of  $p_k^2$  is positive for all  $k$ . Moreover, in this case the contour  $C$  can be deformed to a  $\cup$  shaped contour which goes in the region where  $\operatorname{Re} p^2 > 0$  along the contour. For example, it can go from  $-\infty + i(\beta\mu + \varepsilon)$  to  $-\pi + i(\beta\mu + \varepsilon)$ , then from  $-\pi + i(\beta\mu + \varepsilon)$  to 0, then from 0 to  $\pi + i(\beta\mu + \varepsilon)$  and finally from  $\pi + i(\beta\mu + \varepsilon)$  to  $\infty + i(\beta\mu + \varepsilon)$ ; with  $\varepsilon$  an infinitesimal parameter. For  $\mu > 0$  the function  $h_f$  is given by the same formula

$$h_f(t, \beta\mu) = \frac{1}{2\pi} \int_C dp \, p \tan\left(\frac{p - i\beta\mu}{2}\right) \exp(-tp^2) \quad (5.34)$$

with a contour described above.

## 6 Asymptotics of Quantum Heat Traces

Now, by substituting the heat kernel ansatz (3.21) into the quantum heat traces (5.21) one can compute the integral over  $t$  to obtain the Mellin-Barnes representation of the quantum heat traces

$$\begin{aligned} \Theta_{b,f}(\beta, \mu) &= 2(4\pi)^{-(n+1)/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \, \Gamma(-q) \Gamma[-q + (n+1)/2] \left(\frac{\beta}{2}\right)^{2q-n} \\ &\quad \times F_{b,f}(n-2q, \beta\mu) A_q, \end{aligned} \quad (6.1)$$

where  $c < 0$  and

$$F_b(s, \beta\mu) = \sum_{k=1}^{\infty} \frac{e^{k\beta\mu}}{k^s}, \quad (6.2)$$

$$F_f(s, \beta\mu) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{e^{k\beta\mu}}{k^s}. \quad (6.3)$$

For  $\mu < 0$  these series converge for any  $s$  and define entire functions of  $s$ . For  $\text{Re } s > 0$  they are given by the integrals

$$F_b(s, \beta\mu) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^{t-\beta\mu} - 1}, \quad (6.4)$$

$$F_f(s, \beta\mu) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^{t-\beta\mu} + 1}, \quad (6.5)$$

Indeed, this can be proved by an expansion of the denominator and computing the standard integral canceling the gamma function. The analytic continuation for  $\text{Re } s > -N$  with any positive integer  $N \in \mathbb{Z}_+$  is provided by the integration by parts

$$F_f(s, \beta\mu) = \frac{1}{\Gamma(s+N)} \int_0^\infty dt t^{s+N-1} (-\partial_t)^N \frac{1}{e^{t-\beta\mu} + 1}, \quad (6.6)$$

and similarly for  $F_b$ .

Also, it is easy to see that in the limit  $\mu \rightarrow 0^-$  these functions are determined by the Riemann zeta function

$$F_b(s, 0) = \zeta(s), \quad (6.7)$$

$$F_f(s, 0) = (1 - 2^{1-s}) \zeta(s). \quad (6.8)$$

Note that the fermionic function  $F_f(s, 0)$  is entire but the bosonic function  $F_b(s, 0)$  is meromorphic with a single pole at  $s = 1$ .

For  $\mu > 0$  these series diverge for any  $s$ , however, they can be analytically continued to the polylogarithm,  $\text{Li}_s(z)$ , [14, 12]

$$F_b(s, \beta\mu) = \text{Li}_s(e^{\beta\mu}), \quad F_f(s, \beta\mu) = -\text{Li}_s(-e^{\beta\mu}). \quad (6.9)$$

The integral for  $F_b(s, \beta\mu)$  converges for  $\text{Re } s > 0$  for any  $\mu$  not lying on the positive real axis and defines an analytic function of  $\mu$  with infinitely many cuts along the horizontal lines  $\beta\mu = x + 2\pi in$ , with  $x \geq 0$  and  $n \in \mathbb{Z}$ , including the positive real axis. So, it has a jump across the positive real axis. Therefore,  $F_b(s, \beta\mu)$  is not well defined for real  $\mu > 0$ .

The analytic structure of the function  $F_f(s, \beta\mu)$  is similar. It is an analytic function of  $\mu$  with infinitely many cuts along the horizontal lines  $\beta\mu = x + \pi i(2n+1)$  with  $x \geq 0$ ,  $n \in \mathbb{Z}$ . Therefore, the function  $F_f(s, \beta\mu)$  is an analytic function of  $\mu$  in the neighborhood of the real axis for  $\text{Re } s \geq 0$ . So, it is well defined for real

positive  $\mu$ . Therefore, for any real  $\mu$  the fermionic function  $F_f(s, \beta\mu)$  is an entire function of  $s$ .

For small  $\mu$ , to be precise for  $|\beta\mu| < 2\pi$ , there is an expansion [14]: for non-integer  $s \neq 1, 2, 3, \dots$ ,

$$F_b(s, \beta\mu) = \Gamma(1-s)(-\beta\mu)^{s-1} + \sum_{k=0}^{\infty} \frac{\zeta(s-k)}{k!} (\beta\mu)^k, \quad (6.10)$$

and for positive integer  $s = j = 1, 2, 3, \dots$

$$F_b(j, \beta\mu) = [\psi(j) + \mathbb{C} - \log(-\beta\mu)] \frac{(\beta\mu)^{j-1}}{(j-1)!} + \sum_{k=0, k \neq j-1}^{\infty} \zeta(s-k) \frac{(\beta\mu)^k}{k!}. \quad (6.11)$$

## 6.1 Negative Chemical Potential

The eq. (4.9) is especially useful for studying the asymptotic expansion as  $\beta \rightarrow 0$ . We assume first that  $\mu < 0$ . Then the integral (4.9) is exactly a Mellin-Barnes integral studied in (4.10), (4.11). The only difference with the relativistic heat trace (4.9) is the presence of the functions  $F_{b,f}$ . Therefore, it can be computed by the same method using Lemma 1. We decompose the heat traces in three parts

$$\Theta_{b,f}(\beta, \mu) = \Theta_{b,f}^{\text{sing}}(\beta, \mu) + \Theta_{b,f}^{\text{loc}}(\beta, \mu) + \Theta_{b,f}^{\text{non-loc}}(\beta, \mu), \quad (6.12)$$

and obtain in even dimension  $n = 2m$ ,

$$\Theta_{b,f}^{\text{sing}}(\beta, \mu) \sim \frac{1}{2}(4\pi)^{-m+1/2} \sum_{k=0}^m (-1)^k \frac{(2m-2k)!}{k!(m-k)!} \frac{1}{\beta^{2m-2k}} F_{b,f}(2m-2k, \beta\mu) A_k, \quad (6.13)$$

$$\begin{aligned} \Theta_{b,f}^{\text{loc}}(\beta, \mu) &\sim (-1)^m \frac{1}{2} (4\pi)^{-m} \sum_{k=0}^{\infty} \frac{k!}{(k+m+1)!(2k+1)!} \beta^{2k+2} \\ &\times F_{b,f}(-2k-2, \beta\mu) A_{k+m+1}, \end{aligned} \quad (6.14)$$

$$\Theta_{b,f}^{\text{non-loc}}(\beta, \mu) \sim -(-1)^m \pi^{-m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!(2k+2m+1)!} \beta^{2k+1} F_{b,f}(-2k-1, \beta\mu) A_{k+m+1/2}, \quad (6.15)$$

and in odd dimension  $n = 2m + 1$ ,

$$\Theta_{b,f}^{\text{sing}}(\beta, \mu) \sim 2(4\pi)^{-m-1} \sum_{k=0}^m (-1)^k \frac{(m-k)!}{k!} \left(\frac{2}{\beta}\right)^{2m-2k+1} F_{b,f}(2m+1-2k, \beta\mu) A_k, \quad (6.16)$$

$$\begin{aligned} \Theta_{b,f}^{\text{loc}}(\beta, \mu) \sim & (-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left(\frac{\beta}{2}\right)^{2k+1} \left\{ 2F'_{b,f}(-2k-1, \beta\mu) \right. \\ & \left. + [\psi(k+m+2) + \psi(k+1) - 2\log(\beta/2)] F_{b,f}(-2k-1, \beta\mu) \right\} A_{k+m+1}, \end{aligned} \quad (6.17)$$

$$\begin{aligned} \Theta_{b,f}^{\text{non-loc}}(\beta, \mu) \sim & -(-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left(\frac{\beta}{2}\right)^{2k+1} \\ & \times F_{b,f}(-2k-1, \beta\mu) A'_{k+m+1}. \end{aligned} \quad (6.18)$$

It is worth mentioning that since the fermionic function  $F_f(s, \beta\mu)$  is entire for any real  $\mu$  the formulas for the fermionic quantum heat trace are valid also for zero and positive chemical potential  $\mu$ .

## 6.2 Zero Chemical Potential

Now, we study the limit of zero chemical potential  $\mu \rightarrow 0^-$ . In this case the functions  $F_{b,f}(s, \beta\mu)$  are given by (6.7) and (6.8). So, we obtain

$$\begin{aligned} \Theta_b(\beta, \mu) = & 2(4\pi)^{-(n+1)/2} \frac{1}{2\pi i} \int_{C_0} dq \Gamma(-q) \Gamma[-q + (n+1)/2] \left(\frac{\beta}{2}\right)^{2q-n} \\ & \times \zeta(n-2q) A_q, \end{aligned} \quad (6.19)$$

and

$$\begin{aligned} \Theta_f(\beta, \mu) = & 2(4\pi)^{-(n+1)/2} \frac{1}{2\pi i} \int_{C_0} dq \Gamma(-q) \Gamma[-q + (n+1)/2] \left(\frac{\beta}{2}\right)^{2q-n} \\ & \times (1 - 2^{1-n+2q}) \zeta(n-2q) A_q, \end{aligned} \quad (6.20)$$

Of course, we could have obtained the same results from eqs. (2.28)-(2.29) and (4.7).

First of all, we recall that the fermionic function  $F_f(s, 0)$  is entire. Therefore, there are no additional singularities in the integral and we can just substitute  $\mu = 0$  in the formulas of the previous section. We use the same decomposition of the quantum heat traces (6.12) and the fact that the values of the Riemann zeta function at negative even integers vanish [12]

$$\zeta(-2k) = 0, \quad k = 1, 2, 3, \dots \quad (6.21)$$

Therefore, the function  $\Theta_f^{\text{loc}}(\beta, 0)$  does not contribute to the asymptotic expansion in even dimension  $n = 2m$

$$\Theta_f^{\text{loc}}(\beta, 0) \sim 0. \quad (6.22)$$

For the other parts we obtain in even dimension  $n = 2m$ ,

$$\begin{aligned} \Theta_f^{\text{sing}}(\beta, 0) &\sim \frac{1}{2}(4\pi)^{-m+1/2} \sum_{k=0}^m (-1)^k \frac{(2m-2k)!}{k!(m-k)!} \frac{1}{\beta^{2m-2k}} \\ &\times (1 - 2^{2k-2m+1}) \zeta(2m-2k) A_k, \end{aligned} \quad (6.23)$$

$$\begin{aligned} \Theta_f^{\text{non-loc}}(\beta, 0) &\sim -(-1)^m \pi^{-m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!(2k+2m+1)!} \beta^{2k+1} \\ &\times (1 - 2^{2k+2}) \zeta(-2k-1) A_{k+m+1/2}, \end{aligned} \quad (6.24)$$

and in odd dimension  $n = 2m + 1$ ,

$$\begin{aligned} \Theta_f^{\text{sing}}(\beta, 0) &\sim 2(4\pi)^{-m-1} \sum_{k=0}^m (-1)^k \frac{(m-k)!}{k!} \left(\frac{2}{\beta}\right)^{2m-2k+1} \\ &\times (1 - 2^{2k-2m}) \zeta(2m-2k+1) A_k, \end{aligned} \quad (6.25)$$

$$\begin{aligned} \Theta_f^{\text{loc}}(\beta, 0) &\sim (-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left(\frac{\beta}{2}\right)^{2k+1} \\ &\times \left\{ [\psi(k+m+2) + \psi(k+1) - 2 \log(\beta/2)] (1 - 2^{2k+2}) \zeta(-2k-1) \right. \\ &\left. + 2(1 - 2^{2k}) \zeta'(-2k-1) - 2^{2k+1} \zeta(-2k-1) \log 2 \right\} A_{k+m+1}, \end{aligned} \quad (6.26)$$

$$\begin{aligned} \Theta_f^{\text{non-loc}}(\beta, 0) &\sim -(-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{(\beta/2)^{2k+1}}{k!(k+m+1)!} \\ &\times (1 - 2^{2k+2}) \zeta(-2k-1) A'_{k+m+1}. \end{aligned} \quad (6.27)$$

All we need to do now is to compute the values of the zeta function and its derivatives at negative integer points. It is well known that the values of the zeta function at negative integers are determined by the Bernoulli numbers [12]

$$\zeta(-2k+1) = (-1)^k \frac{2(2k-1)!}{(2\pi)^{2k}} \zeta(2k) = -\frac{B_{2k}}{2k}. \quad (6.28)$$

Also, by using the functional equation for the zeta function we obtain the derivatives at negative integer points

$$\zeta'(-2k) = (-1)^k \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k+1) \quad (6.29)$$

$$\zeta'(-2k+1) = -(-1)^k \frac{2(2k-1)!}{(2\pi)^{2k}} \{ \zeta'(2k) + [\psi(2k) - \log(2\pi)] \zeta(2k) \} \quad (6.30)$$

Now we turn to the bosonic case. In this case the function  $F_b(s, 0) = \zeta(s)$  acquires an additional pole at  $s = 1$ . This leads to an additional pole of the integrand at  $q = (n-1)/2$  which needs to be taken into account when computing the contour integral (6.19). In even dimensions this pole is simple, however, in odd dimensions this pole coincides with one of the poles of one of the gamma functions, making it a double pole.

Therefore our Lemma 1 needs a modification. We consider the cases of even and odd dimensions separately. In even dimension  $n = 2m$  we still have

$$\Theta_b^{\text{loc}}(\beta, 0) \sim 0 \quad (6.31)$$

and we just need to compute an extra term due to an additional simple pole. The extra term is easily computed by the residue calculus. We obtain

$$\Theta_b(\beta, 0) = \Theta_b^{\text{sing}}(\beta, 0) + S(\beta) + \Theta_b^{\text{non-loc}}(\beta, 0), \quad (6.32)$$

where

$$S(\beta) = (-1)^m \pi^{-m} \frac{m!}{(2m)!} \frac{1}{\beta} A_{m-1/2} \quad (6.33)$$

is the additional term. The other terms are:

$$\begin{aligned} \Theta_b^{\text{sing}}(\beta, 0) &\sim \frac{1}{2} (4\pi)^{-m+1/2} \sum_{k=0}^m (-1)^k \frac{(2m-2k)!}{k!(m-k)!} \frac{1}{\beta^{2m-2k}} \\ &\quad \times \zeta(2m-2k) A_k, \end{aligned} \quad (6.34)$$

$$\begin{aligned} \Theta_b^{\text{non-loc}}(\beta, 0) &\sim -(-1)^m \pi^{-m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!(2k+2m+1)!} \beta^{2k+1} \\ &\quad \times \zeta(-2k-1) A_{k+m+1/2}. \end{aligned} \quad (6.35)$$

In odd dimension  $n = 2m + 1$  the zeta function has a pole at  $q = m$  which coincides with a pole of one of the gamma functions making it a double pole. So, we need to remove it from the previous sum and add the recomputed correct term. This will be the terms proportional to  $A_m$  and  $A'_m$ . We obtain

$$\Theta_b(\beta, 0) = \tilde{\Theta}_b^{\text{sing}}(\beta, 0) + \tilde{S}(\beta) + \Theta_b^{\text{loc}}(\beta, 0) + \Theta_b^{\text{non-loc}}(\beta, 0), \quad (6.36)$$

where

$$\tilde{\Theta}_b^{\text{sing}}(\beta, 0) \sim 2(4\pi)^{-m-1} \sum_{k=0}^{m-1} (-1)^k \frac{(m-k)!}{k!} \left(\frac{2}{\beta}\right)^{2m-2k+1} \zeta(2m+1-2k) A_k, \quad (6.37)$$

$$\begin{aligned} \Theta_b^{\text{loc}}(\beta, 0) \sim & (-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left(\frac{\beta}{2}\right)^{2k+1} \\ & \times \left\{ [\psi(k+m+2) + \psi(k+1) - 2\log(\beta/2)] \zeta(-2k-1) \right. \\ & \left. + 2\zeta'(-2k-1) \right\} A_{k+m+1}, \end{aligned} \quad (6.38)$$

$$\Theta_b^{\text{non-loc}}(\beta, 0) \sim -(-1)^m 2(4\pi)^{-m-1} \sum_{k=0}^{\infty} \frac{1}{k!(k+m+1)!} \left(\frac{\beta}{2}\right)^{2k+1} \zeta(-2k-1) A'_{k+m+1}. \quad (6.39)$$

Here we removed the last term from the sum for  $k = m$  in eq. (6.37) since it would be singular as  $\zeta(1)$  and recomputed it. The correct term is

$$\tilde{S}(\beta) = -2 \frac{(-1)^m}{m!} (4\pi)^{-m-1} \frac{1}{\beta} \left\{ A'_m - [\psi(m+1) - \psi(1)] A_m + 2\log(\beta/2) A_m \right\} \quad (6.40)$$

Note that the extra term is proportional to  $\beta^{-1}$  and  $\beta^{-1} \log \beta$  and is determined by the invariant  $A_{(n-1)/2}$  and its derivative  $A'_{(n-1)/2}$  at the point  $q = (n-1)/2$ . This invariant is like the zeta regularized determinant (which is proportional to  $A_{n/2}$  and  $A'_{(n-1)/2}$ ); it is global and cannot be computed locally in a generic case. For the lack of a better name we call this new invariant the *residue*.

## 7 Conclusion

The primary goal of this paper was to introduce and to study some new invariants of second-order elliptic partial differential operators on manifolds,  $\Theta_{b,f,r}(\beta, \mu)$

given by (2.11) and (2.19)-(2.20) that we call the relativistic heat trace and the quantum heat traces. Of special interest are the asymptotics of these invariants as  $\beta \rightarrow 0$ . First, we showed how these heat traces can be reduced to the some integrals of the standard heat trace. Then, by using a special Mellin-Barnes representation of the heat kernel for the Laplace type operator introduced in our paper [1] we were able to compute the asymptotics of the quantum heat traces. We showed that the asymptotic expansion has both power and logarithmic terms. We expressed the coefficients of the asymptotic expansion in terms of values of an entire function  $A_q$  and its derivatives at integer and half-integer points. The values of this entire function at the positive integer points  $A_k$  are equal to the standard heat trace coefficients and are locally computable, while the values of the function at half-integer points,  $A_{k+1/2}$ , as well as the derivatives of it at integer points,  $A'_k$ , are non-trivial global invariants.

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